Expected Value of the Sample Variance

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January 25, 2020

Recall that the variance of a random variable X with mean μ is defined as $\sigma^2 = \mathrm{Var}[X] = \mathrm{E}[(X-\mu)^2] = \mathrm{E}[X^2] - \mu^2$. Given i.i.d. samples $X_1, \ldots, X_n$ from the distribution of X, we estimate σ^2 by $s_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \mu_n)^2$, where $\mu_n = \frac{1}{n}\sum_{i=1}^n X_i$ is the usual estimator of the mean μ. We show that s_n^2 is an *unbiased* estimator of σ^2, in that $\mathrm{E}[s_n^2] = \sigma^2$.

To simplify things, note that the variance of a random variable X is unchanged if we subtract a constant c: $\mathrm{Var}[X-c] = \mathrm{Var}[X]$. We can choose $c = \mu$, and hence can assume without loss of generality that $\mathrm{E}[X] = 0$. It follows that

- $\mathrm{E}[X_i] = 0$ for $i = 1, 2, \ldots, n$;
- $\mathrm{E}[X_i^2] = \mathrm{E}[X_i^2] - 0 = \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2 = \sigma^2$ for $i = 1, 2, \ldots, n$; and
- $\mathrm{E}[X_i X_j] = \mathrm{E}[X_i] \cdot \mathrm{E}[X_j] = 0 \cdot 0 = 0$ for $1 \le i \ne j \le n$,

where we have used the fact that the X_i's are mutually independent.

Next observe that

$$\begin{aligned}
\mathrm{E}[\mu_n^2] &= \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2\right] = \mathrm{E}\left[\frac{1}{n^2}\sum_{i=1}^n X_i^2 + \frac{2}{n^2}\sum_{i \ne j} X_i X_j\right] \\
&= \frac{1}{n^2}\sum_{i=1}^n \mathrm{E}[X_i^2] + \frac{2}{n^2}\sum_{i \ne j}\mathrm{E}[X_i X_j] = \frac{1}{n^2}(n\sigma^2) + \frac{2}{n^2}\cdot 0 = \frac{\sigma^2}{n}.
\end{aligned}$$

Putting everything together and noticing that $\sum_{i=1}^n X_i = n\mu_n$, we have

$$\begin{aligned}
\mathrm{E}[s_n^2] &= \mathrm{E}\left[\frac{1}{n-1}\sum_{i=1}^n (X_i - \mu_n)^2\right] = \frac{1}{n-1}\,\mathrm{E}\left[\sum_{i=1}^n (X_i^2 - 2X_i\mu_n + \mu_n^2)\right] \\
&= \frac{1}{n-1}\left(\sum_{i=1}^n \mathrm{E}[X_i^2] - 2\,\mathrm{E}\left[\mu_n \sum_{i=1}^n X_i\right] + n\,\mathrm{E}[\mu_n^2]\right) \\
&= \frac{1}{n-1}\left(\sum_{i=1} \mathrm{E}[X_i^2] - 2n\,\mathrm{E}[\mu_n^2] + n\,\mathrm{E}[\mu_n^2]\right) = \frac{1}{n-1}\left(n\sigma^2 - n\,\mathrm{E}[\mu_n^2]\right) \\
&= \frac{1}{n-1}\left(n\sigma^2 - n\cdot\frac{\sigma^2}{n}\right) = \sigma^2.
\end{aligned}$$